Analytic Number Theory Sheet 2

Lent Term 2020

1. For a Dirichlet series $F(s)=\sum \frac{a_n}{n^s}$ let σ_c have the property that $F(s)$ converges for all s with $\sigma > \sigma_c$ and for no s with $\sigma < \sigma_c$. Let σ_a have the property that $F(s)$ converges *absolutely* if $\sigma > \sigma_a$ and does not converge absolutely if $\sigma < \sigma_a$. Show that
$$\sigma_c \le \sigma_a \le \sigma_c + 1.$$
Give examples to show that both $\sigma_c = \sigma_a$ and $\sigma_c + 1 = \sigma_a$ are possible.

2. For fixed $\sigma \in \mathbb{R}$ let $\nu(\sigma)$ denote the infimum of those exponents ν such that $\zeta(\sigma+it) \ll |t|^\nu$ for all $|t| \ge 4$. (The Lindelöf hypothesis is the conjecture that $\nu(1/2)=0$.)
   (a) Show that $\nu(\sigma)=0$ for $\sigma \ge 1$.
   (b) Show that $\nu(\sigma) \le 1-\sigma$ for $0<\sigma\le 1$.
   (c) Show that $\nu(\sigma)=\nu(1-\sigma)+1/2-\sigma$, and in particular $\nu(\sigma)=1/2-\sigma$ for $\sigma \le 0$. (You may use Stirling's approximation, that $|\Gamma(s)| \asymp t^{\sigma-1/2}e^{-\pi t/2}$ as $t\to\infty$ for σ uniformly bounded.)

3. Show that
$$\sum_{\substack{a,b\ge 1\\(a,b)=1}} \frac{1}{a^2b^2} = \frac{5}{2}.$$
Hint: Use the fact that $\sum_{d|n}\mu(d)=0$ *if* $n>1$.

4. Sketch a proof that if $s\ne 1$ and $\zeta(s)\ne 0$ then, if x is not an integer,
$$\sum_{n\le x}\frac{\Lambda(n)}{n^s} = \frac{x^{1-s}}{1-s} - \lim_{T\to\infty}\sum_{\substack{\rho\\|\gamma|\le T}}\frac{x^{\rho-s}}{\rho-s} - \frac{\zeta'}{\zeta}(s) + \sum_{k=1}^{\infty}\frac{x^{-2k-s}}{2k+s}.$$

5. (a) Using elementary methods show that
$$\sum_{n\le x}\Lambda(n)\left\lfloor\frac{x}{n}\right\rfloor = x\log x + O(x).$$
   (b) Deduce that
$$\sum_{n\le x}\frac{\Lambda(n)}{n} \sim \log x.$$
   Compare this to the result of Question 4 as $s\to 1$.

6. (a) Show that if $\sigma>0$ and $k\ge 1$ then
$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{y^s}{s^{k+1}}\,\mathrm{d}s = \begin{cases}\frac{(\log y)^k}{k!} & \text{if } y\ge 1 \text{ and}\\ 0 & \text{if } y \le 1.\end{cases}$$
   (b) Give an explicit formula for any $k\ge 1$ for
$$\sum_{n\le x}\Lambda(n)\left(\log\frac{x}{n}\right)^k,$$
   sketching a proof of the formula you give.