Analytic Number Theory Sheet 1

Lent Term 2019

1. Let $\tau_k(n) = \sum_{a_1 \cdots a_k = n} 1$, so that for example $\tau_2(n) = \tau(n)$. Prove that

$$\sum_{n \le x} \tau_k(n) = x P_k(\log x) + O(x^{1-1/k} (\log x)^{k-2})$$

where P is a polynomial with degree $k-1$ and leading coefficient $1/(k-1)!$.

2. Let $\omega(n)$ count the number of distinct prime divisors of n.

   (a) Prove that

$$\sum_{n \le x} \omega(n) = x \log\log x + O(x).$$

   (b) Prove the 'variance bound'

$$\sum_{n \le x} |\omega(n) - \log\log x|^2 \ll x \log\log x.$$

   (c) Deduce that

$$\sum_{n \le x} |\omega(n) - \log\log n|^2 \ll x \log\log x.$$

   and hence 'almost all n have $(1+o(1))\log\log n$ distinct prime divisors' in the sense that the number of $n \le x$ such that $|\omega(n) - \log\log n| > (\log\log n)^{3/4}$ is $o(x)$.

3. (a) Show that

$$\sum_{n \le x} \frac{1}{n} = \log x + \gamma - \frac{\{x\} - 1/2}{x} + O(x^{-2}).$$

   (b) Let $\Delta(x)$ be the error term in the approximation for the sum of the divisor function, so that

$$\sum_{n \le x} \tau(n) = x \log x + (2\gamma - 1)x + \Delta(x).$$

   We proved in lectures that $\Delta(x) = O(x^{1/2})$. Show that

$$\int_0^x \Delta(t)\,\mathrm{d}t \ll x.$$

4. Prove, for all n, the more precise upper bound

$$\tau(n) \le n^{\frac{\log 2 + o(1)}{\log\log n}}.$$

By considering n of the form $\prod_{p \le z} p$, show that this upper bound is sharp.

5. (a) Show that

$$\gamma = -\int_0^\infty e^{-t} \log t\,\mathrm{d}t.$$

(b) Let

$$A(x) = \sum_{p \le x} \frac{1}{p} = \log\log x + c + E(x),$$

say, where $E(x) \ll 1/\log x$, and c is some constant. Show that for $\delta > 0$

$$\sum_p \frac{1}{p^{1+\delta}} = \delta \int_2^\infty \frac{\log\log t + c}{t^{1+\delta}}\,\mathrm{d}t + \delta \int_2^\infty \frac{E(t)}{t^{1+\delta}}\,\mathrm{d}t.$$

(c) Show that

$$\sum_p \frac{1}{p^{1+\delta}} + \log\delta - c + \gamma = \delta \int_2^\infty \frac{E(t)}{t^{1+\delta}}\,\mathrm{d}t - \delta \int_1^2 \frac{\log\log t + c}{t^{1+\delta}}\,\mathrm{d}t.$$

(d) Deduce that

$$\sum_p \frac{1}{p^{1+\delta}} + \log\delta \to c - \gamma$$

as $\delta \to 0$.

(e) Deduce that

$$\sum_{p \le x} \frac{1}{p} = \log\log x + \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p} \right) + o(1)$$

and hence that

$$\prod_{p \le x} \left(1 - \frac{1}{p}\right)^{-1} \sim e^\gamma \log x.$$

6. (Challenging, for the keen) Let $r \ge 0$ be fixed. (Recall that $f \ll_r g$ means $|f(x)| \le C(r)\,|g(x)|$ for all large enough x, and for some constant $C(r) > 0$ which may depend on r.)

(a) Prove that

$$\sum_{p \le x} \tau(p)^r \log p \ll_r x.$$

(b) Prove that

$$\sum_p \sum_{k \ge 2} \frac{\tau(p^k)^r k \log p}{p^k} \ll_r 1.$$

(c) Using $1 \star \Lambda = \log$, deduce that

$$\sum_{n \le x} \tau(n)^r \log n \ll_r x \sum_{n \le x} \frac{\tau(n)^r}{n}.$$

(d) Using $\log x = \log(x/n) + \log n$, deduce that

$$\sum_{n \le x} \tau(n)^r \ll_r \frac{x}{\log x} \sum_{n \le x} \frac{\tau(n)^r}{n}.$$

(e) Finally, through bounding the sum on the right hand side in (d) by a product over primes, deduce that

$$\sum_{n \le x} \tau(n)^r \ll_r x(\log x)^{2^r - 1}.$$

7. (For the very keen) Fill in the details of the 14 point plan of the elementary proof of the prime number theorem given in the printed lecture notes.