

Primitive Representations by Three Coprime Bases

Let $a, b, c > 1$ be pairwise coprime. For $D \geq 0$, put

$$\mathcal{L}_D = \{a^i b^j c^k : i, j, k \geq 0, i + j + k = D\}.$$

Divisibility is coordinatewise, so every subset of $\mathcal{L}_D$ is primitive. It is therefore enough to construct intervals of ordinary subset sums on one level.

Theorem 1. *Every sufficiently large integer is the sum of a primitive finite subset of $\{a^i b^j c^k : i, j, k \geq 0\}$.*

After permuting the bases, assume throughout that

$$a < c < b.$$

The proof rests on the following source of arithmetic progressions.

Lemma 2 (Progressions on an edge). *There is an integer H , depending only on c , such that for every $L > 1$ there are $\delta \geq 0$, integers s, d with $d > 0$, and sets $E_0, \dots, E_{L-1} \subset \mathcal{L}_\delta$ satisfying*

$$\sum_{x \in E_r} x = s + rd \quad (0 \leq r < L),$$

while the exponent of b in every member of every E_r is at most H .

Proof. Let $\phi = \varphi(c)$ and $H = \phi(c - 2)$. For $E \geq H$ and $0 \leq r < c$, set

$$P_E(r) = \{a^{E-H+\phi t} b^{\phi(c-2-t)} : 0 \leq t < r\}.$$

These are subsets of $\mathcal{L}_E$, their b exponents are at most H , and

$$\sum_{x \in P_E(r)} x \equiv ra^{E-H} \pmod{c}.$$

For a word $\omega = (r_0, \dots, r_{n-1}) \in \{0, \dots, c-1\}^n$, take

$$F(\omega) = \bigcup_{i=0}^{n-1} c^i P_{H+n-1-i}(r_i) \subset \mathcal{L}_{H+n-1}.$$

Successive reduction modulo $c, c^2, \dots, c^n$ recovers the letters of ω ; hence the sums of the $F(\omega)$ run bijectively through the residues modulo c^n . Since $a < c$ and the b exponents are bounded, all these sums are $O(c^n)$. Color each residue by the quotient of its corresponding sum upon division by c^n . The number of colors is bounded independently of n . For large n , van der Waerden's theorem gives a progression of L residues with one color, and the corresponding sums form the required progression. $\square$

We shall amplify one such progression. We use two elementary facts. First, if $A < B$ are coprime and $L = 4AB$, then

$$\left\{ \sum_{e=0}^N r_e A^{N-e} B^e : 0 \leq r_e < L \right\} \quad (1)$$

contains an interval $[U_N, V_N]$ such that

$$V_N - U_N \geq 2AB^{N+1}, \quad V_N \leq K_1 B^{N+1}, \quad (2)$$

where K_1 is independent of N . This is the usual radix induction: choose the last digit modulo A and treat the remaining quotient at the preceding stage.

Second, fix a modulus Δ . There is K_2 such that, for every large D and every residue ρ modulo Δ , one can find a set $Q_\rho \subset \mathcal{L}_D$ whose members lie on the faces of the exponent simplex and satisfy

$$\sum_{x \in Q_\rho} x \equiv \rho \pmod{\Delta}, \quad \sum_{x \in Q_\rho} x \leq K_2 c^D. \quad (3)$$

Indeed, for a fixed large R , the three face monomials

$$b^R c^{D-R}, \quad a^R c^{D-R}, \quad a^{D-R} b^R$$

have common divisor 1, so their residues generate $\mathbb{Z}/\Delta\mathbb{Z}$. Every residue is a nonnegative combination with coefficients below Δ . Eventual periodicity modulo Δ supplies distinct copies by moving a common period along a face, and $a < c$ bounds their total by a constant times c^D .

Lemma 3 (A full scale). *There is an integer $C > 1$ such that, for every sufficiently large M , the subset sums of some level $\mathcal{L}_{D_M}$ contain every integer in*

$$[C^M, C^{M+1}).$$

Proof. Choose an integer $u > H + 1$, then choose $v > 0$ so large that

$$ba^v < c^v.$$

Put

$$A = a^{u+v}, \quad B = b^u c^v, \quad q = abc.$$

Then $A < B$ and $(A, B) = 1$. Apply Lemma 2 with $L = 4AB$, obtaining δ, s, d and the sets E_r , and put $\Delta = qd$.

Fix an auxiliary integer N . For digits $r_0, \dots, r_N$, multiply E_{r_e} by $qA^{N-e}B^e$ and take the union. All terms have degree

$$D_N = \delta + (u + v)N + 3.$$

If $a^i b^j c^k \in E_{r_e}$, then $0 \leq j \leq H$, and the corresponding term $qA^{N-e}B^e a^i b^j c^k$ has b exponent $ue + j + 1$. Hence the e th block lies in the range $[ue + 1, ue + H + 1]$; these ranges are disjoint because $u > H + 1$. Their sum is

$$O_N + \Delta \sum_{e=0}^N r_e A^{N-e} B^e, \quad O_N = qs \sum_{e=0}^N A^{N-e} B^e.$$

Thus (1) and (2) give a represented progression $O_N + \Delta[U_N, V_N]$ on the single level $\mathcal{L}_{D_N}$.

Use (3) with this Δ and set $Z_N = K_2 c^{D_N}$. Every integer in

$$J_N = [O_N + \Delta U_N + Z_N, O_N + \Delta V_N]$$

is represented: choose a face correction with the required residue modulo Δ , subtract it, and use the progression. The blocks lie in the interior because of the factor q , so they are disjoint from the correction. Moreover

$$\frac{c^{D_N}}{B^N} = c^{\delta+3} \left(\frac{c}{b}\right)^{uN} \longrightarrow 0.$$

For all large N , the left endpoint of J_N is at most KB^{N+1} for some constant K , and J_N has length at least $X := qB^N$.

It remains to gain an unbounded factor in the length. For each s with $H + 2 \leq s \leq N$, inspect the interior points of $\mathcal{L}_{D_N}$ having b exponent $uN + s + 1$. Along this line successive values have ratio c/a . The value nearest the a axis is at most X for large N , since

$$\frac{b^s a^{vN+\delta-s}}{c^{vN}} \leq a^\delta \left(\frac{ba^v}{c^v}\right)^N,$$

whereas the value nearest the c axis is larger than X , since its ratio to X is $c^\delta (b/c)^s > 1$. Hence the line contains a term z_s with

$$\frac{a}{c}X < z_s \leq X.$$

The terms z_s are distinct, lie above all blocks in b exponent, and lie off the faces. They may therefore be adjoined independently to every representation from J_N . Since each z_s is no larger than the length of J_N , the standard overlap argument extends J_N to a represented interval I_N with the same left endpoint and

$$\sup I_N - \inf I_N \geq \sum_{s=H+2}^N z_s \geq \frac{a}{c}(N - H - 1)qB^N.$$

Thus, for some $\kappa > 0$ and all large N ,

$$\inf I_N \leq KB^{N+1}, \quad \sup I_N - \inf I_N \geq \kappa NB^N.$$

Choose an integer t with $KB^{1-t} \leq 1$, and set $C = B$. Given large M , apply the construction with $N = M - t$. Then

$$\inf I_N \leq B^M = C^M,$$

and, since $\kappa(M - t) \geq B^{t+1}$ for large M ,

$$\sup I_N \geq \sup I_N - \inf I_N \geq B^{M+1} = C^{M+1}.$$

Hence I_N contains $[C^M, C^{M+1})$. Taking $D_M = D_{M-t}$ proves the lemma. $\square$

Proof of Theorem 1. Every sufficiently large integer lies in $[C^M, C^{M+1})$ for some sufficiently large M , and hence is a subset sum of $\mathcal{L}_{D_M}$. Such a subset is primitive. $\square$